METRIC SPACES: FINAL EXAM 2012

DOCENT: A. V. KISELEV

Evaluation: $\min\Big(100\%, \max\big(5\,\text{prb} \times 20\% \cdot \big[{}^{1.00}_{1.15^{\text{top}}}\big]\big)\Big)$.

Problem 1. Let $(\mathcal{X}, \mathrm{d})$ be a non-empty metric space, r and s be two positive radii, and $B_r^{\mathrm{d}}(x) = B_s^{\mathrm{d}}(y)$ for some $x, y \in \mathcal{X}$.

- Is it true that $r = s$?
- Is it true that $x = y$?

Problem 2. Let $(\mathcal{X}, \mathrm{d})$ be a non-empty metric space. By definition, for all $x, y \in \mathcal{X}$ put

$$\mathrm{d}_1\colon \mathcal{X} \times \mathcal{X} \longmapsto [0,1), \qquad \mathrm{d}_1(x,y) = \frac{\mathrm{d}(x,y)}{1 + \mathrm{d}(x,y)}, \qquad \text{and}$$

$$\mathrm{d}_2\colon \mathcal{X} \times \mathcal{X} \longmapsto [0,1], \qquad \mathrm{d}_2(x,y) = \min\big(1, \mathrm{d}(x,y)\big).$$

Show that the functions d_1 and d_2 are also metrics on $\mathcal{X}$.

Problem 3. Let $A, B \subseteq \mathbb{E}^n$ be two subsets and consider their sum

$$A + B = \big\{x + y \mid x \in A,\ y \in B\big\}.$$

Suppose that A is open and B is closed.

- Is it true that $A + B$ is open?
- Is it true that $A + B$ is closed?

Problem 4. Let $(\mathcal{X}, \mathrm{d})$ be a metric space and $\big\{A_i \mid i \in \mathcal{I}\big\}$ be a family of connected subsets $A_i \subseteq \mathcal{X}$ such that $A_i \cap A_j \neq \varnothing$ for all indexes $i, j \in \mathcal{I}$. Prove that the union $A = \bigcup_{i \in \mathcal{I}} A_i$ is connected.

Problem 5. Prove that the algebraic equation $7x = 1 - x^5$ has a unique solution in the segment $[0,1] \subset \mathbb{R}$.

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GOOD LUCK!